

On Local Domain Symmetry for Model Expansion

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Intro

General symmetry definition

Given a vocabulary Σ , theory T , and domain D , a **symmetry** σ for T is a permutation on **the set of D, Σ -structures Γ_D** such that for all $I \in \Gamma_D$:

$$I \models T \text{ iff } \sigma(I) \models T$$

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Why study symmetry?

speeding up search – **symmetry breaking**

avoid parts of the search space symmetrical to failed parts

Outline

Intro

Theory symmetry

MX symmetry

Efficient breaking

More symmetry

Conclusion

Prelims: First-Order Logic (FO)

- vocabulary Σ of (function and predicate) symbols S/k
- Σ -theory T
- Σ -structure I
 - domain D
 - interpretations S^I for all $S \in \Sigma$

Semantics captured by satisfiability relation:

$$I \models T$$

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In ASP: program \leftrightarrow theory, set of facts \leftrightarrow structure.

For the rest of the talk: vocabulary and domain are implicit and fixed.

Running example: graph coloring

T_{gc} :

$$\forall x_1 y_1: Edge(x_1, y_1) \Rightarrow Color(x_1) \neq Color(y_1)$$

$$\forall x_2 y_2: Edge(x_2, y_2) \Rightarrow V(x_2) \wedge V(y_2)$$

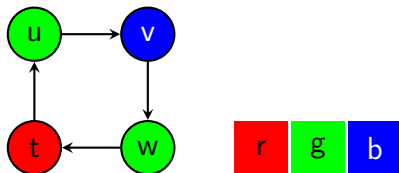
$$\forall x_3: C(Color(x_3))$$

I_{gc} :

$$V^{I_{gc}} = \{t, u, v, w\} \quad C^{I_{gc}} = \{r, g, b\}$$

$$Edge^{I_{gc}} = \{(t, u), (u, v), (v, w), (w, t)\}$$

$$Color^{I_{gc}} = t \mapsto r, u \mapsto g, v \mapsto b, w \mapsto g, r \mapsto r, g \mapsto g, b \mapsto b$$



Note: $I_{gc} \models T_{gc}$

Symmetry for a theory

General symmetry definition

Given a vocabulary Σ , theory T , and domain D , a **symmetry** σ for T is a permutation on **the set of D, Σ -structures Γ_D** such that for all $I \in \Gamma_D$:

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Symmetries compose to symmetry **groups**

Global Domain Symmetry

Permutation π on D induces permutation σ_π on Γ_D :

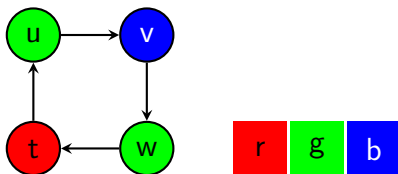
$$\begin{aligned}(\pi(d_1), \dots, \pi(d_n)) \in P^{\sigma_\pi(I)} &\text{ iff } (d_1, \dots, d_n) \in P^I \\ f^{\sigma_\pi(I)}(\pi(d_1), \dots, \pi(d_n)) = \pi(d_0) &\text{ iff } f^I(d_1, \dots, d_n) = d_0\end{aligned}$$

Let's call such induced σ_π a **Global Domain Symmetry** for T .
Intuitively, domain renaming π preserves satisfiability:

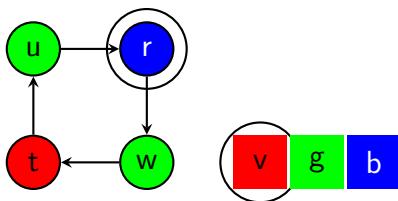
$$\sigma_\pi(I) \models T \text{ iff } I \models T$$

Graph coloring ctd.

Let $\pi = (v r)$, so σ_π maps



to



which still models T_{gc} .

Connectively closed argument positions

More precise notion of domain symmetry:

apply π only on limited set of argument positions A .

- **Argument position** $S|n$ with $S/k \in \Sigma$ and $1 \leq n \leq k$ denotes S 's n^{th} argument. $f|0$ is output argument position.

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- Argument positions are **connected** under theory T if one occurs as subterm of the other, if they are connected by $=$, or if they are connected by quantified variables.

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- Argument positions are **connected** under theory T if one occurs as subterm of the other, if they are connected by $=$, or if they are connected by quantified variables.
- A set of argument positions A is **connectively closed** under T if no other argument positions of Σ are connected to A under T .

Intuitively, a partition of connectively closed argument positions under T corresponds to a well-defined **typing** of T .

Graph coloring ctd.

T_{gc} :

$$\forall x_1 y_1: Edge(x_1, y_1) \Rightarrow Color(x_1) \neq Color(y_1)$$

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Connectively closed argument position partition under T_{gc} :

- $\{V|1, Edge|1, Edge|2, Color|1\}$
- $\{C|1, Color|0\}$

Local Domain Symmetry

Permutation π on D and argument position set A induce permutation σ_π^A on Γ_D :

$$\begin{aligned}(\pi^A(d_1), \dots, \pi^A(d_n)) \in P^{\sigma_\pi^A(I)} &\text{ iff } (d_1, \dots, d_n) \in P^I \\ f^{\sigma_\pi^A(I)}(\pi^A(d_1), \dots, \pi^A(d_n)) = \pi^A(d_0) &\text{ iff } f^I(d_1, \dots, d_n) = d_0\end{aligned}$$

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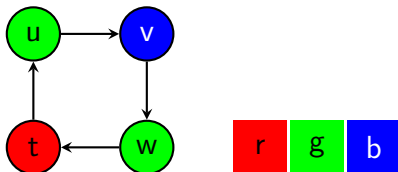
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If A is connectively closed under T , σ_π^A is a **local domain symmetry** of T .

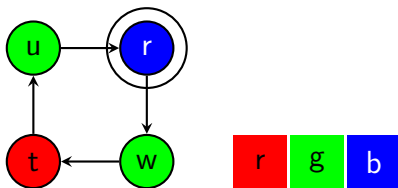
Intuitively, σ_π^A permutes domain element tuples according to some type A of T .

Graph coloring ctd.

Let $\pi = (v r)$ and $A = \{V|1, Edge|1, Edge|2, Color|1\}$, so σ_π^A maps



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Local Domain Symmetry

- So far, so good: more or less known in literature (MACE, SEM, Paradox, ...)

Local Domain Symmetry

- So far, so good: more or less known in literature (MACE, SEM, Paradox, ...)
- What if we have to take pre-interpreted symbols into account? → [Model eXpansion](#)

First-Order Model Expansion (MX)

In:

- vocabulary $\Sigma = \Sigma_{in} \cup \Sigma_{out}$ ($\Sigma_{in} \cap \Sigma_{out} = \emptyset$)
- Σ -theory T
- Σ_{in} -structure I_{in}
 - domain D
 - interpretations $S^{I_{in}}$ to $S \in \Sigma_{in}$

Out:

- Σ_{out} -structure I_{out} such that $I_{in} \sqcup I_{out} \models T$
 - same domain D
 - $I_{in} \sqcup I_{out}$ merges both structures to a Σ -structure
- or "unsat"

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Shortened as $MX(T, I_{in})$.

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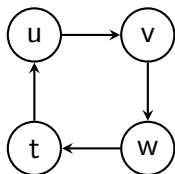
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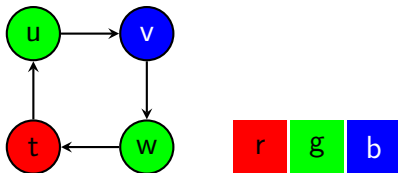


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I_{gcout} :

$$Color^{I_{gcout}} = t \mapsto r, u \mapsto g, v \mapsto b, w \mapsto g, r \mapsto r, g \mapsto g, b \mapsto b$$

$I_{gcin} \sqcup I_{gcout}$:



Note: $I_{gcin} \sqcup I_{gcout} \models T_{gc}$

Symmetry for MX

Symmetry for $MX(T, I_{in})$

A symmetry σ for $MX(T, I_{in})$ is a permutation on the set of D, Σ_{out} -structures Γ_D such that for all $I \in \Gamma_D$:

$$I_{in} \sqcup I_{out} \models T \quad \text{iff} \quad I_{in} \sqcup \sigma(I_{out}) \models T$$

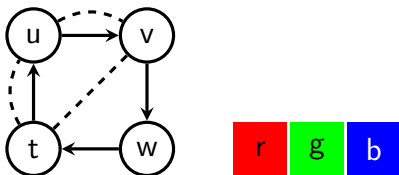
Intuitively, symmetry must preserve input structure.

Sufficient condition for MX-symmetry?

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- Local domain symmetry σ_{π}^A where A is connectively closed under T and π such that $\sigma_{\pi}^A(l_{in}) = l_{in}$
- Does not work well for independent symbols connected under T :



Both graphs $Edge/2$ and $Edge'/2$ have different symmetries, but since their argument positions typically are connected by a $V/1$ argument position, **no π exists** that is consistent with the symmetry of both graphs.

Sufficient condition for MX-symmetry?

We defined transformation for $MX(T, I_{in})$ to $MX(T^*, I_{in}^*)$ such that

Sufficient condition for MX-symmetry

σ_π^A is a local domain symmetry for $MX(T, I_{in})$ if A is connectively closed under T^* and $\sigma_\pi^A(I_{in}^*) = I_{in}^*$.

Intuitively, $MX(T, I_{in})$ **decouples** all occurrences of pre-interpreted symbols.

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Answer:

- **generate-and-test** for domain element swaps $(d_1 d_2)$

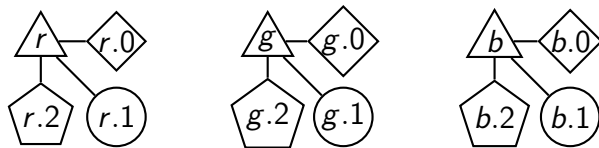
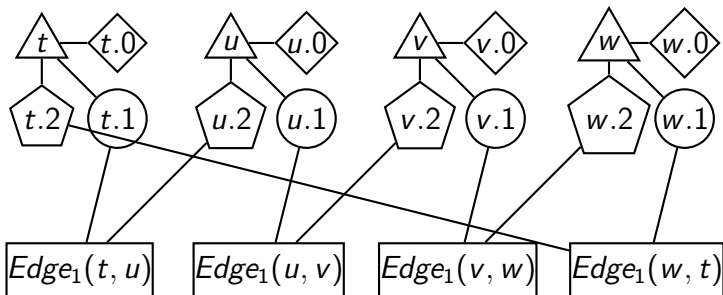
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Answer:

- **generate-and-test** for domain element swaps (d_1 d_2)
- encode to **graph automorphism problem** for more complicated π
 - In: I_{in}, A
 - Out: generators π that induce symmetry σ_{π}^A
 - Size of graph **depends on size of I_{in}** , not on size of T

Graph coloring ctd.



Symmetry breaking

Given symmetry group \mathbb{G} , construct **symmetry breaking formula** $\varphi(\mathbb{G})$.

$\varphi(\mathbb{G})$ is **sound** if for each I_{out} , there exists some $\sigma \in \mathbb{G}$ such that $I_{in} \sqcup \sigma(I_{out}) \models \varphi(\mathbb{G})$.

$\varphi(\mathbb{G})$ is **complete** if for each I_{out} , there exists exactly one $\sigma \in \mathbb{G}$ such that $I_{in} \sqcup \sigma(I_{out}) \models \varphi(\mathbb{G})$.

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What is the size of $\varphi(\mathbb{G})$ to break \mathbb{G} completely?

Breaking local domain symmetry completely

\mathbb{G}_δ^A is a **local domain interchangeability group** if A is a set of argument positions, $\delta \subseteq D$, and each permutation π over δ induces a local domain symmetry σ_π^A .

\mathbb{G}_δ^A is broken completely with $O(|\delta|^2)$ sized symmetry breaking formula if **A contains at most one argument position $S|i$ for each symbol $S \in \Sigma_{out}$.**

Intuitively, after grounding, \mathbb{G}_δ^A represents row interchangeability of a Boolean variable matrix. Ordering the rows breaks all symmetry.

Graph coloring ctd.

Let $A = \{C|1, Color|0\}$, $\delta = \{r, b, g\}$. \mathbb{G}_δ^A represents the **interchangeability of colors**, and can be broken completely with a small symmetry breaking formula.

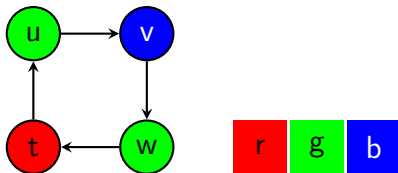
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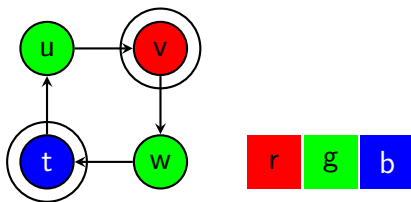
However, e.g. for the Ramsey number problem, $A = \{Edge|1, Edge|2, Color|1, \dots\}$, so A' does not satisfy the **“one argument position”** condition.

Symmetry notions not captured by local domain symmetry

E.g. swapping colors of node v and t :



to



which still models T_{gc} .

Conclusion

- Notion of **local domain symmetry**
- **Sufficient condition** for symmetry detection in the context of an input structure
- **Symmetry detection approach** on predicate level
- **Completeness guarantee** for symmetry breaking
- **Limits** of our approach

- Notion can be extended to aggregates, non-monotonic rules, etc.
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Future work

Extend local domain symmetry to capture other notions of symmetry.

Thanks for your attention!

Questions?